VIBRATIONS OF A CANTILEVERED PIEZOCERAMIC PLATE WITH A CORRUGATED INTERMEDIATE LAYER

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In the present study, calculations are performed to obtain the period-mean Lagrangians of the plane and flexural vibrations of a cantilevered three-layer rectangular piezoelectric resonator with a corrugated metallic intermediate layer. The finite elements method is used to calculate the natural frequencies.

A significant improvement in the characteristics of electromechanical transducers can be achieved through the use of elements with different properties. For example, the efficiency of a transducer whose operation is based on the transverse piezoelectric effect can be increased if an electrically passive corrugated layer is included in it. However, no rigorous theoretical calculations have yet been completed for such a resonator.

In the present study, we propose an approximate method of determining the natural and forced vibrations of a cantilevered three-layer rectangular piezoelectric ceramic transducer with a corrugated metal intermediate layer (liner).

We will examine a rectangular three-layer sandwich. The two outermost layers are made of a thickness-polarized piezoelectric ceramic whose front surfaces have been covered with electrodes. We assume that a thin metal plate has been placed in the middle of the sandwich. The sinusoidal plate has been corrugated along one side and is rigidly attached to the piezoceramic layers. The entire sandwich is secured in cantilever fashion, as shown in Fig. 1.

We will refer the transducer to a Cartesian coordinate system $O\xi\eta\zeta$, having chosen the $O\xi$ and $O\eta$ axes to be in the middle plane of the transducer. The $O\zeta$ axis coincides with the direction of polarization of the piezoceramic plates (see Fig. 1). Let the transducer have the dimensions $a \times b \times h$. The thicknesses of the piezoceramic layers are the same and are equal to $2h_1$, while the overall height of the corrugated liner is equal to $2h_0$. Thus, $h = 2(h_0) + 2(2h_1)$.

We introduce dimensionless coordinates by means of the formulas

$$\xi = ax,\ \eta = ay,\ \zeta = hz$$

We use $u_x^{(j)}(x, y, z)$, $u_y^{(j)}(x, y, z)$, $u_z^{(j)}(x, y, z)$ to represent the displacements in the j-th layer and $\varphi^{(k)}(x, y, z)$ $(k = 1, 3)$ to represent the potential in the outmost layers (the superscript $j = 1$ denotes the bottom layer, $j = 2$ denotes the middle layer, and $j = 3$ denotes the top layer).

Assuming the thickness h of the transducer to be small compared to the characteristic dimensions a and b and taking the outermost piezoceramic layers to be stiff relative to the corrugated liner, we adopt the following hypotheses:

1) the normal stresses $\sigma_{zz}^{(j)}$ are much lower than the stresses $\sigma_{xx}^{(j)}$ and $\sigma_{yy}^{(j)}$, $j = 1, 2, 3$;

2) cylindrical bending takes place, i.e., all of the functions are independent of y and $u_y^{(j)} = 0$, $j = 1, 2, 3$;

3) the displacements $u_z^{(j)}$ are the same for the entire sandwich and depend only on the coordinate x, i.e., $u_z^{(j)} = \tilde{w}(x)$, $j = 1, 2, 3$;

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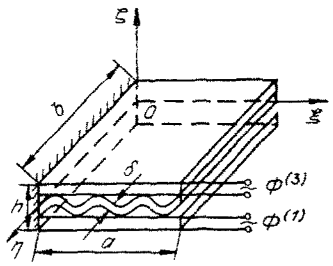

Fig. 1

4) the displacements $u_x^{(j)}$ are different in each layer and obey the broken-normal hypothesis [1]:

$$u_x^{(j)} = u_{cx}^{(j)} + z_j\psi_j, \quad j = 1, 2, 3.$$

Here, $u_{cx}^{(j)} = u_{cx}^{(j)}(x)$ are the displacements of the middle planes of the j-th layers in the direction of the x axis; z_j is the z coordinate of the local coordinate system with its origin at the middle of the j-th layer; $\psi_1 = \psi_3 = -\tilde{w}_{,x}/a$ is the angle of rotation of the normal in the piezoceramic layers; ψ_2 is a function of x;

5) the component of electric induction $D_x^{(k)}$ along the x axis is negligibly small, i.e., $D_x^{(k)} = 0$, k = 1, 3;

6) in accordance with [2], the intervening corrugated layer can be replaced by an equivalent orthotropic plate whose effective characteristics are found from the condition of equality of the displacements and angles of rotation with the application of identical forces and moments to suitably cut-out elements.

If the scheme normally employed for multilayered piezoceramic plates [3, 4] is used here, the above-formulated hypotheses make it possible to determine the stress and strain components simply in terms of the functions $u_{cx}^{(1)}$, $u_{cx}^{(3)}$, and $\tilde{w}$.

Since the effective characteristics of the orthotropic plate are different in tension and bending, the displacement fields should also be divided into parts describing tension and bending. This can be done by introducing the functions

$$u = u_{cx}^{(1)} + u_{cx}^{(3)}, \quad v = u_{cx}^{(1)} - u_{cx}^{(3)}.$$

Then the function u will describe tension of the transducer, while the functions v and $\tilde{w}$ will describe its bending.

Now we assume that vibrations of the transducer is caused by potential differences $\Phi^{(1)}$ and $\Phi^{(3)}$ between the top and bottom electrodes of the bottom (1) and top (3) layers, respectively. Meanwhile, $\Phi^{(1)}$ and $\Phi^{(3)}$ change in accordance with a harmonic law exp(iωt). Thus, we are studying a problem concerning the steady-state vibration of a transducer. As is often done for such problems, we will use the same notation for the amplitude functions. For example, if $\Phi^{(1)} = \tilde{\Phi}^{(1)}\exp(i\omega t)$, then instead of $\tilde{\Phi}^{(1)}$ we will use the notation $\Phi^{(1)}$, etc.

In accordance with the foregoing, we should separate the above overall problem on steady-state vibration into problems concerning plane vibration (p) and flexural vibration (f). After completing several transformations, we can obtain formulas for the period-averaged kinetic K_p, K_f and potential W_p, W_f energies of the transducer for plane and flexural vibrations:

$$K_p = \varkappa\Omega^2 \int_0^1 \gamma_{k1}^{p} u^2\, dx,$$

$$K_f = \varkappa\Omega^2 \int_0^1 \left(\gamma_{k1}^{f} v^2 + 2\gamma_{k2}^{f} v w_{,x} + \gamma_{k3}^{f} w_{,x}^2 + \gamma_{k4}^{f} w^2\right) dx,$$

$$W_p = \varkappa \int_0^1 \left(\gamma_{w1}^{f} u_{,x}^2 + \Phi_p u_{,x}\right) dx,$$

$$W_{f} = \varkappa \int_0^1 (\gamma^{f}_{w1} v^1_{,x} + \gamma^{f}_{w2} v^2 + 2\gamma^{f}_{w3} v_{,x} w_{,xx} + 2\gamma^{f}_{w4} v w_{,x} + \gamma^{f}_{w5} w^2_{,xx} + \gamma^{f}_{w6} w^2_{,x} + \Phi_{f} v_{,x}) dx,$$

$$\varkappa = \frac{bhc_{11}}{4a};\ \Omega = \omega a \sqrt{\frac{\rho_k}{c_{11}}};\ w = \frac{\tilde{w}}{a};$$

$$\Phi_{p} = \tilde{a}\frac{B_{32}}{c_{11}}(\Phi^{(1)} + \Phi^{(3)});\ \Phi_{f} = \tilde{a}\frac{B_{32}}{c_{11}}(\Phi^{(1)} - \Phi^{(3)});$$

$$\gamma^{p}_{k1} = 2\tilde{h}_1 + \tilde{h}_0\tilde{\rho}_c;$$

$$\gamma^{f}_{k1} = 2\tilde{h}_1 + \frac{1}{3}\tilde{h}_0\tilde{\rho}_c \quad \gamma^{f}_{k2} = -\frac{2}{3}\tilde{h}_0\tilde{h}_1\tilde{\rho}_c;$$

$$\gamma^{f}_{k3} = \frac{4}{3}\tilde{h}_1^2\gamma^{p}_{k1}; \qquad \gamma^{f}_{k4} = 4\tilde{a}^2\gamma^{p}_{k1};$$

$$\gamma^{p}_{w1} = 2\tilde{h}_1 Q_{11} + \tilde{h}_0 H^{p}_{11},$$

$$\gamma^{f}_{w1} = 2\tilde{h}_1 Q_{11} + \frac{1}{3}\tilde{h}_0 H^{f}_{11};\quad \gamma^{f}_{w2} = \frac{1}{2}\tilde{a}\frac{a}{h_0}a^{f}_{55};$$

$$\gamma^{f}_{w3} = -\frac{2}{3}\tilde{h}_0\tilde{h}_1 H^{f}_{11}; \qquad \gamma^{f}_{w4} = -\tilde{a}^2\left(1 + \frac{h_1}{h_0}\right)a^{f}_{55};$$

$$\gamma^{f}_{w5} = \frac{4}{3}\tilde{h}_1^2(2\tilde{h}_1 Q_{11} + \tilde{h}_0 H^{f}_{11});\ \gamma^{f}_{w6} = 2\tilde{h}_0\left(1 + \frac{h_1}{h_0}\right)^2\tilde{a}^2 a^{f}_{55};$$

$$\tilde{a} = \frac{a}{h};\quad \tilde{h}_0 = \frac{h_0}{h};\ \tilde{h}_1 = \frac{h_1}{h};$$

$$Q_{11} = (R_{11} + B^2_{32}/B_{31})/c_{11};$$

$$a^{f}_{55} = A^{f}_{55}/c_{11};\quad \tilde{\rho}_c = \rho_{0c}/\rho_k;$$

$$R_{11} = c_{11} - c^2_{13}/c_{33};$$

$$B_{31} = \text{э}_{33} + e^2_{33}/c_{33};\quad B_{32} = e_{31} - e_{33}c_{13}/c_{33};$$

$$H^{p}_{11} = D^{p}_{11}/c_{11};\quad H^{f}_{11} = D^{f}_{11}/c_{11};$$

$$D^{p}_{11} = A^{p}_{11} - (A^{p}_{13})^2/A^{p}_{33};\quad D^{f}_{11} = A^{f}_{11} - (A^{f}_{13})^2/A^{f}_{33}.$$

In the above formulas, c_{lm}, э_{lm} and e_{lm} are standard [5] notations for the elastic moduli, permittivities, and piezoelectric constants, respectively; ρ_k is the density of the piezoceramic; A_{lm} are the elastic moduli of the orthotropic material [6]; D_{lm} are the stiffnesses of the orthotropic material for a plane stress state. The superscripts p and f with A_{lm} and D_{lm} denote characteristics in tension and bending. The moduli are different for these problems and have the form

$$D^{p}_{11} = \frac{E}{k_2 D},\ D^{p}_{13} = \nu D^{p}_{11},\ D^{p}_{33} = \frac{k_1 E}{D},$$

$$D^{f}_{11} = \frac{E}{k_3 D},\ D^{f}_{13} = \nu D^{f}_{11},\ D^{f}_{33} = \frac{k_4 E}{D},\ D^{f}_{55} = A^{f}_{55} = E/k_0.$$

Here,

$$D = 1 - \frac{\nu^2}{k_1 k_2};\ k_0 = \frac{1}{2\pi\varepsilon}\sqrt{1 - k^2}\left[K(k) + \frac{1}{\varepsilon^2\gamma^2}(K(k) - (1 - 4\gamma^2)E(k))\right];$$

$$k_1 = \frac{2\varepsilon}{\pi}\sqrt{1 - k^2}K(k);$$

$$k_2 = \frac{2}{\pi\varepsilon}\left[\sqrt{1 - k^2}K(k) + \frac{1}{\varepsilon^2 k^2\sqrt{1 - k^2}}((1 - k^2)K(k) + (2k^2 - 1)E(k))\right];$$

$$k_3 = \frac{2}{\pi\varepsilon^3}\sqrt{1 - k^2}K(k);\ k_4 = \varepsilon^4 k_2;\ \varepsilon = \frac{\delta}{2h_0};\ k^2 = \frac{\gamma^2}{1 + \gamma^2};\ \gamma = \frac{2\pi h_0}{l};$$

where K(k) and E(k) are first- and second-order complete elliptic integrals, respectively; l is the length of one period of the sinusoidal corrugation; δ is its thickness; E and ν are the elastic modulus and Poisson's coefficient of the material of the corrugation.

Finally,

$$\rho_{0c} = \frac{\delta E(k)}{\pi h_0 \sqrt{1-k^2}} \rho c$$

(ρ_c is the density of the material of the corrugation).

It should be noted that a change in the direction of the corrugation of the middle layer in the direction along the y axis is accompanied by transposition of the coefficients $D^{p,f}_{11}$ and $D^{p,f}_{13}$, while A^f_{55} is set equal to $E/(2(1 + \nu))$. All of the other relations remain valid in this case.

To obtain the equations of motion, it is convenient at this point to resort to the principle of virtual displacements for steady-state vibrations:

$$\delta L_{p,f} = 0,$$

where $L_{p,f} = K_{p,f} - W_{p,f}$. The variations δu, δv, and δw must conform to the principal boundary conditions

$$u = 0, \quad v = 0, \quad w = 0, \quad w_{,x} = 0, \quad x = 0.$$

Thus, we will have linear unidimensional boundary-value problems for both plane and flexural vibrations of the transducer. Such problems can be solved in closed form. To find the natural frequencies in the respective problems, it is sufficient to set Φ_p and Φ_f equal to zero and solve eigenvalue problems.

The natural frequencies ω^p_m (m = 1, 2, ...) of the plane vibrations will be determined by the formula

$$\omega^p_m = \frac{\pi}{a} \sqrt{\frac{c_{11}\gamma^p_{w1}}{\rho_k \gamma^p_{k1}}} \left(m - \frac{1}{2}\right), \quad m = 1, 2, \ldots$$

However, the exact solutions obtained in closed form for the problem of flexural vibration are quite cumbersome and are not completely satisfactory from a computational standpoint. This has to do with the fact that — as is usually the case in dynamic problems of electroelasticity [5] — it is necessary to determine complex roots of a bicubic equation, suitably normalize complex-valued hyperbolic functions, and find the zeros of a sixth-order complex determinant. Thus, the computer program that was developed to obtain an exact solution has proven unsatisfactory. It has been determined to be much more efficient and accurate to use the finite elements method (FEM) to solve the variational problem $\delta L_f = 0$. Since the given problem requires us to find the extremum of the functional L_f in the class of functions v and w satisfying the conditions

$$v \in W^1_2[0,1], v(0) = 0, \quad w \in W^2_2[0,1], \quad w(0) = w_{,x} = (0),$$

quadratic Langrangian elements for v and cubic Hermitian elements for w [7] are found to be a convenient finite-element approximation. A program written on the basis of these approximations proved to be exceptionally effective both in terms of speed and in regard to accuracy and flexibility. Finding the natural frequencies Ω^f of flexural vibration by the FEM reduces to a generalized eigenvalue problem:

$$[A]\{v\} = (\Omega^f)^2 [B]\{v\}.$$

Here, {v} is the column vector of the "nodal displacements" v, w, and $w_{,x}$; the mass [B] and stiffness [A] matrices are positive-definite and symmetric. The problem was solved numerically on the basis of Householder transforms in combination with the QL-algorithm [8].

Table 1 shows the first four natural frequencies ω^f_m (m = 1, 2, 3, 4) of flexural vibrations of the transducer calculated from programs based on the exact solution and the finite-element approximation (M is the number of finite elements). The initial data for Table 1 was as follows: the material of the piezoceramic layers — ceramic TsTS-19; the

TABLE 1

m	Exact solution	FEM ($M=4$)	FEM ($M=10$)
	$\omega_m^f \cdot 10^{-6}$, rad/sec		
1	0,01794	0,01797	0,01795
2	0,07601	0,07646	0,07604
3	0,1641	0.1659	0,1643
4	0.2602	0,2647	0,2604

TABLE 2

m	N			
	5	10	15	20
	$\omega_m^f \cdot 10^{-6}$, rad/sec			
1	0,01498	0,01644	0,01740	0,01797
2	0,05611	0.06395	0,07093	0,07646
3	0,1167	0,1364	0,1524	0,1659
4	0.1986	0.2210	0,2440	0,2647

material of the corrugated layer — aluminum; a = 0.02 m; h = 0.002 m; $2h_1$ = 0.0005 m; thickness of the sinusoid of the corrugation δ = 0.0001 m; corrugation along the length a; number of corrugations $N = a/l = 20$.

It is evident from Table 1 that the FEM provides very good accuracy in determining the natural frequencies even for a small number of elements. The FEM calculations also require less computer time and are stable against changes in the initial data.

It should be pointed out that the first four natural frequencies of plane vibration ω_m^p are an order of magnitude greater than the corresponding natural frequencies of flexural vibration ω_m^f. Thus, given the same initial data as for Table 1, $\omega_1^p = 0.2405 \cdot 10^6$ rad/sec, $\omega_2^p = 0.7216 \cdot 10^6$ rad/sec. Since the first two natural frequencies of the transducer are of the greatest practical value, the importance of the plane vibrations is minor in the given case.

Table 2 shows the dependence of the natural frequencies ω_m^f of flexural vibration on the number N of corrugations along the cantilever as determined by the FEM with M = 4. It is apparent that an increase in N is accompanied by an increase in the natural frequencies. This tendency can be attributed to an increase in the stiffness of the system.

Thus, the method and programs developed here make it possible to efficiently determine the natural frequencies and other parameters of vibrational processes in a cantilevered rectangular transducer with a corrugated liner. By changing the initial data, it is possible to optimize the parameters of the transducer in order to optimize its characteristics.

The method proposed here can also be generalized to other laminated unidimensional or axisymmetric transducers with soft passive intermediate layers.

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